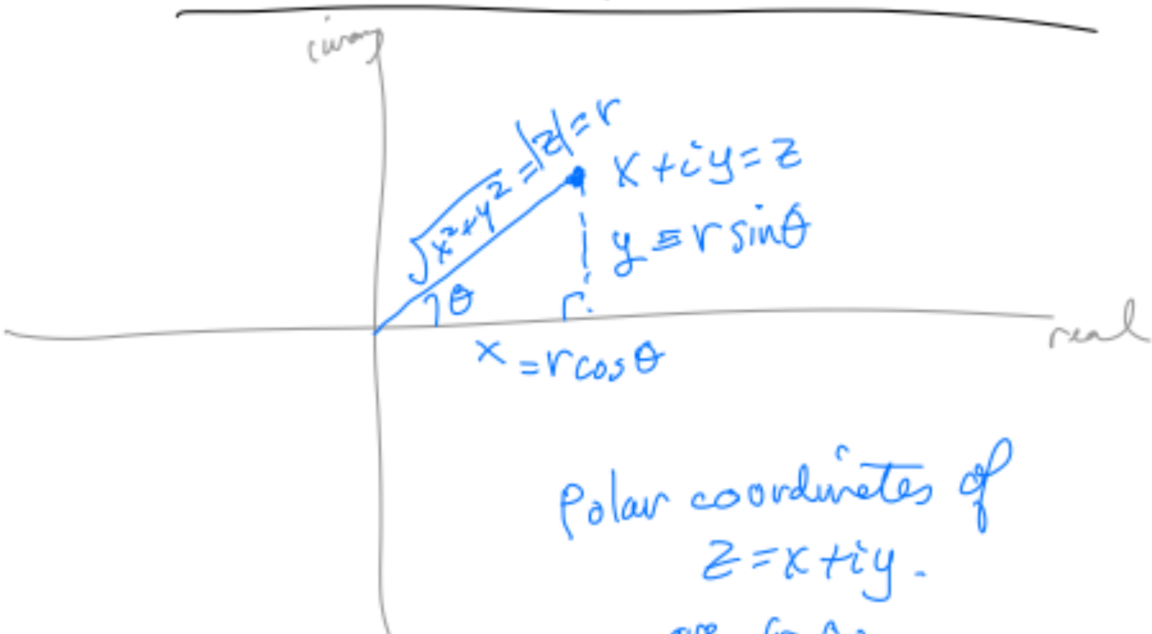
Warmup: Simplify $\dfrac{3-i}{4-2i}$.

Solution: $\dfrac{(3-i)(2+i)}{2\underbrace{(2-i)(2+i)}_{5}} = \boxed{\dfrac{7+i}{10}} = \dfrac{7}{10} + \dfrac{1}{10}i.$

$(2-i)(2+i) = |2+i|^2 = 4+1 = 5$

Geometric Interpretation of Multiplication



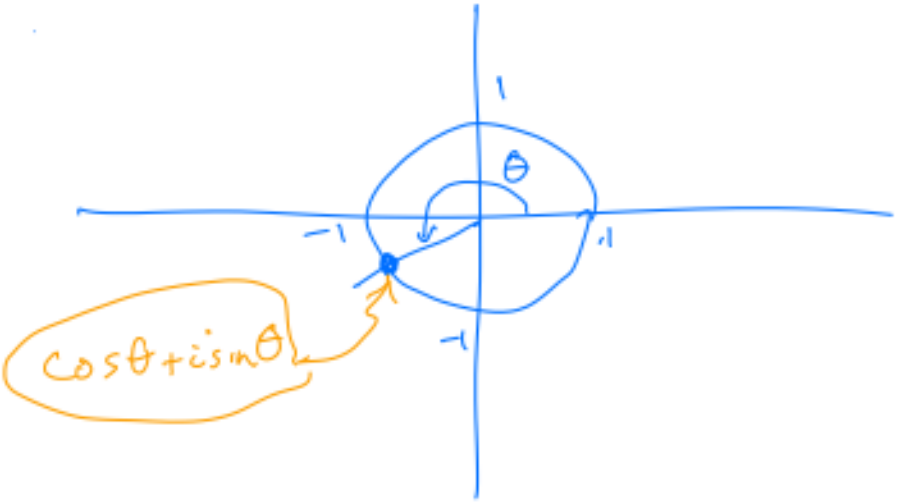
Polar coordinates of $z = x+iy$ are (r,θ)

$r = \sqrt{x^2+y^2}$

$\tan\theta = \dfrac{y}{x}$ if $x \neq 0$.

Polar form of

$$x+iy = (\underbrace{r\cos\theta}_{x}) + i(\underbrace{r\sin\theta}_{y}) = r(\cos\theta + i\sin\theta)$$



Interlude:

Taylor series of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

In Calculus, we can <u>prove</u> that the Taylor series for e^x actually converges to e^x at every $x \in \mathbb{R}$.

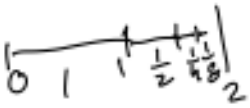
Other examples:

Geometric series

exists for any $x \in \mathbb{R} - \{1\}$

only exists for $x \in (-1, 1)$

$$\frac{1}{1-x} = \underbrace{1 + x + x^2 + x^3 + \cdots}_{\text{Converges to } \frac{1}{1-x}}$$



Converges to $\frac{1}{1-x}$ if & only if $|x| < 1$.

Proof: $$(1 + x + x^2 + x^3 \cdots + x^n)(1-x) = 1 + x + x^2 + \cdots + x^n - x - x^2 - \cdots - x^n - x^{n+1}$$
$$= 1 - x^{n+1}.$$

$$\Rightarrow 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}.$$

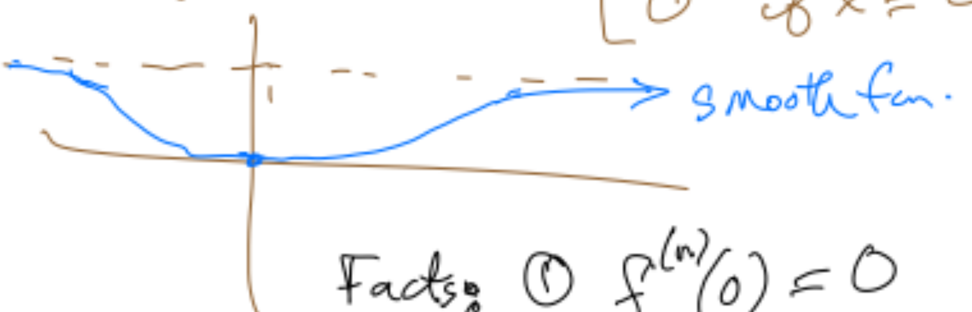
True $\forall x \in \mathbb{R}-\{1\}$

$$\lim_{n\to\infty}\left(1+x+x^2+\dots+x^n\right) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

$$\lim_{n\to\infty} x^{n+1} = \begin{cases} 0 & \text{if } |x|<1 \\ 1 & \text{if } x=1 \\ \text{does not exist} & \text{otherwise.} \end{cases}$$

if $|x|<1 \Rightarrow \lim_{n\to\infty}\left(1+x+x^2+\dots+x^n\right) = \frac{1}{1-x}$

Example: $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0 \end{cases}$



Facts: ① $f^{(n)}(0) = 0$ for all n.

$\therefore$ Taylor series of $f(x)$ is 0.

② $f(x) \neq$ Taylor series of $f(x)$ unless $x=0$.

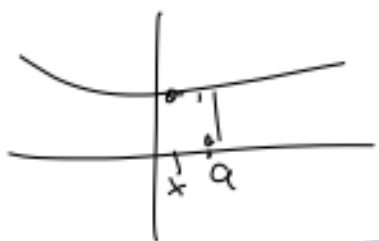
What is always true for smooth fcns?

$f(x) =$ Taylor poly. + remainder.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 +$$

$$\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}}_{\text{Taylor-Lagrange remainder}}$$

where c is between x & a.



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Look what happens when we start using complex numbers: [To convert a real-valued fcn to a complex-valued fcn, we just can insert the complex # into the Taylor series.]

$$e^{i\theta} = e^{(i\theta)} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\Rightarrow \boxed{x + iy = re^{i\theta}}$$

$\theta = \arg(z)$

polar form of a complex #.

$r = |z|$

All the exponential rules work for complex #s!

eg. $e^{z_1+z_2} = e^{z_1} e^{z_2} \quad \forall z_1, z_2 \in \mathbb{C}$.

(Because you can derive $e^{A+B} = e^A e^B$ from the Taylor series of both sides.) $e^{A+B} = 1 + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \cdots$

$= 1 + A + B + \frac{A^2}{2!} + AB + \frac{B^2}{2!} + \cdots$

$= (e^A)(e^B)$.

also $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$.

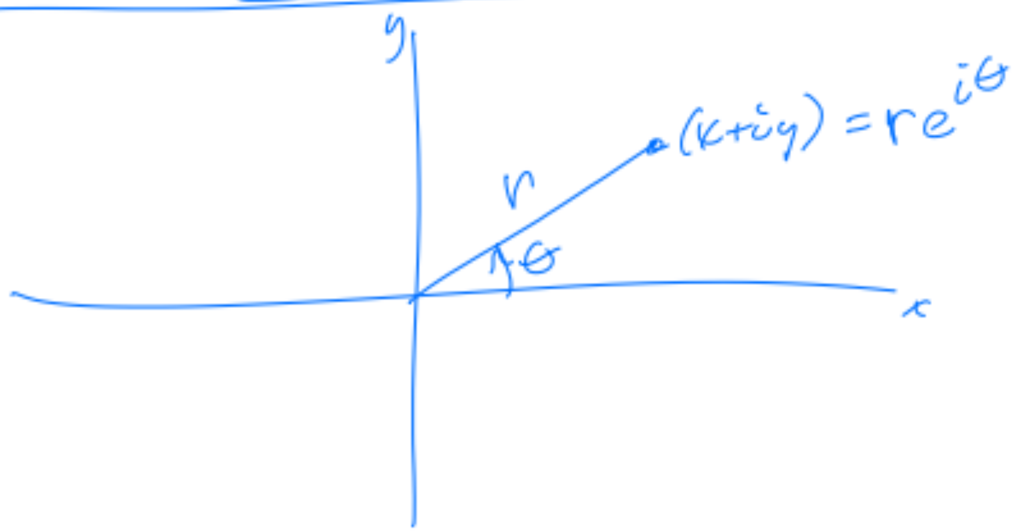
$(e^{z_1})^{z_2} = e^{z_1 z_2}$ (if we figure that out).

<u>Example:</u> $e^{i(2\theta)} = (e^{i\theta})^2$

$\Rightarrow \cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$

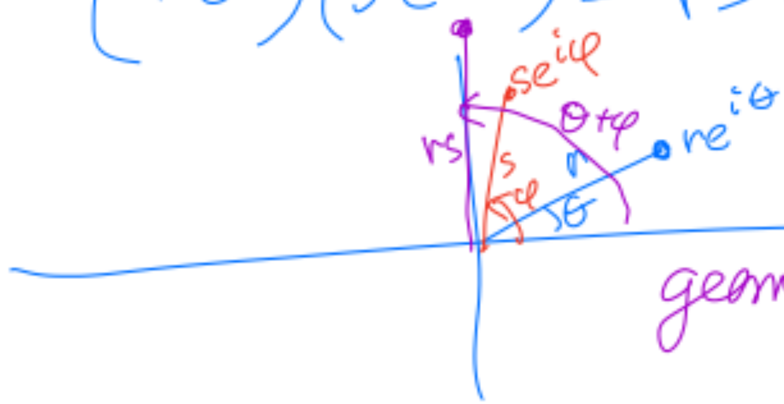
$\cos(2\theta) + i\sin(2\theta) = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$$\Rightarrow \boxed{\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta . \end{aligned}}$$



$$(a+bi)(c+di) = (ac-bd) + (bc+ad)i$$

$$(re^{i\theta})(se^{i\varphi}) = rs\,e^{i(\theta+\varphi)}$$



$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$|z_1 z_2| = |z_1||z_2|$,

geometric interpretation of multiplication

- **More identities**

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}, \quad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$$

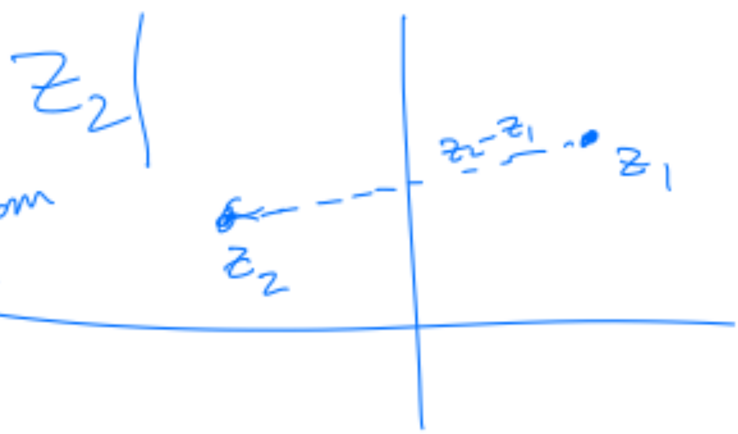
$z = x+iy$
$\bar{z} = x-iy$
$2x$

$$|\mathrm{Re}(z)| \le |z|, \quad |\mathrm{Im}(z)| \le |z|$$

If $z_1, z_2 \in \mathbb{C}$

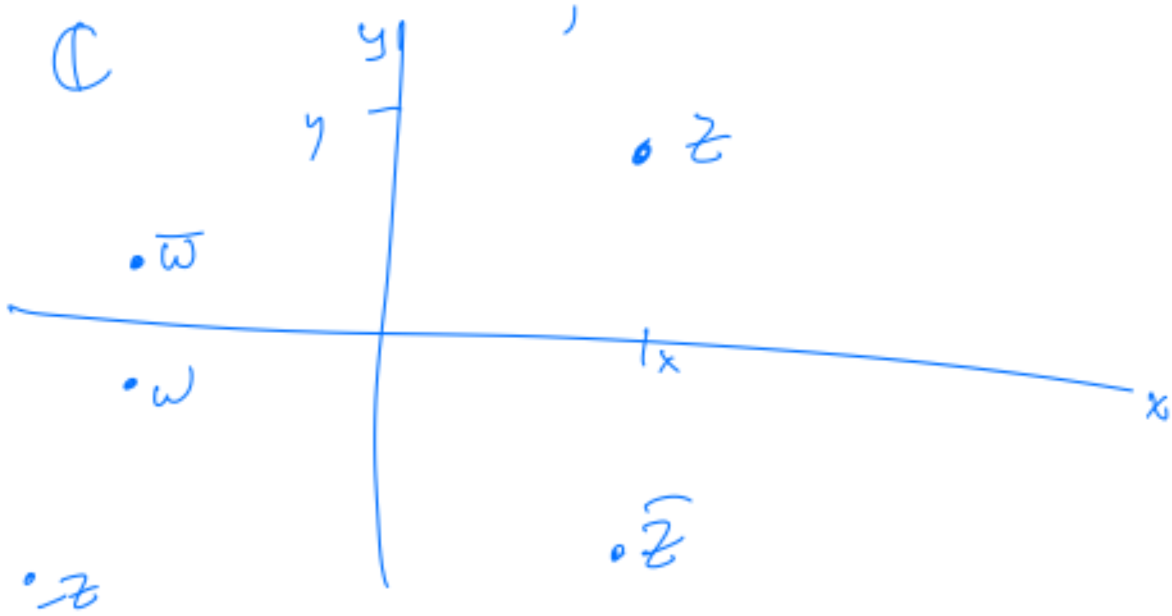
$$|z_1 - z_2|$$

= distance from z_1 to z_2



$|z_1+z_2|$

$\mathbb{C}$



$z \longmapsto \bar{z}$ reflection across the real axis.

Triangle $\leq$s

$$|z_1+z_2| \leq |z_1| + |z_2|$$

$$|z_1-z_2| \geq \big|\,|z_1| - |z_2|\,\big|$$

$$|z_1+z_2| \geq \big|\,|z_1| - |z_2|\,\big|$$

<u>Question</u>: Why don't we have subtraction and division properties?

$$\left.\begin{aligned} A-B &= A+(-B) \\ \frac{A}{B} &= A\cdot\left(\frac{1}{B}\right) = A\cdot B^{-1}. \end{aligned}\right\}$$

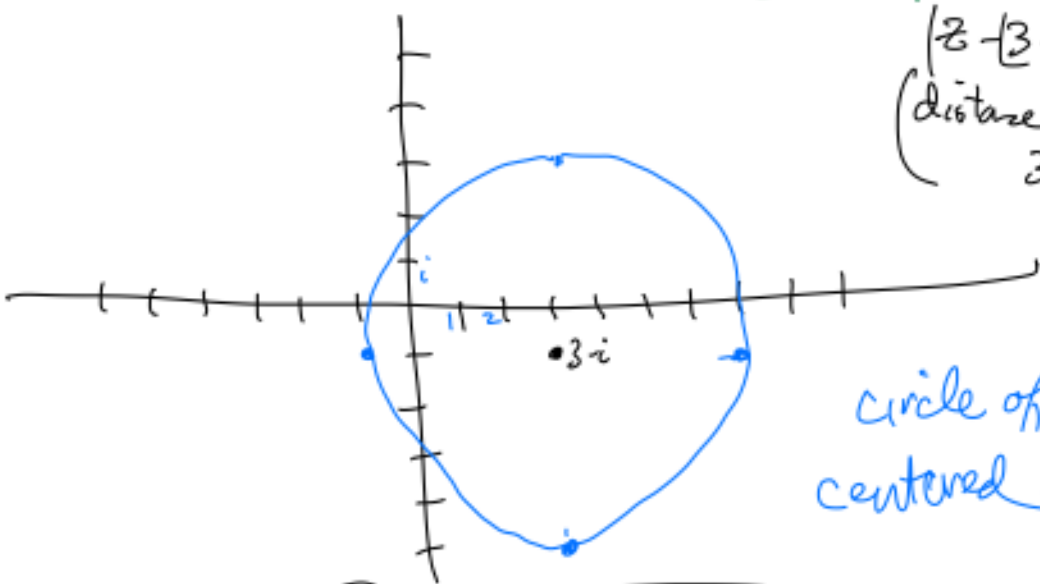
Can derive properties from addition & multiplication properties.

Example: Graph $|z-3+i|=4$.

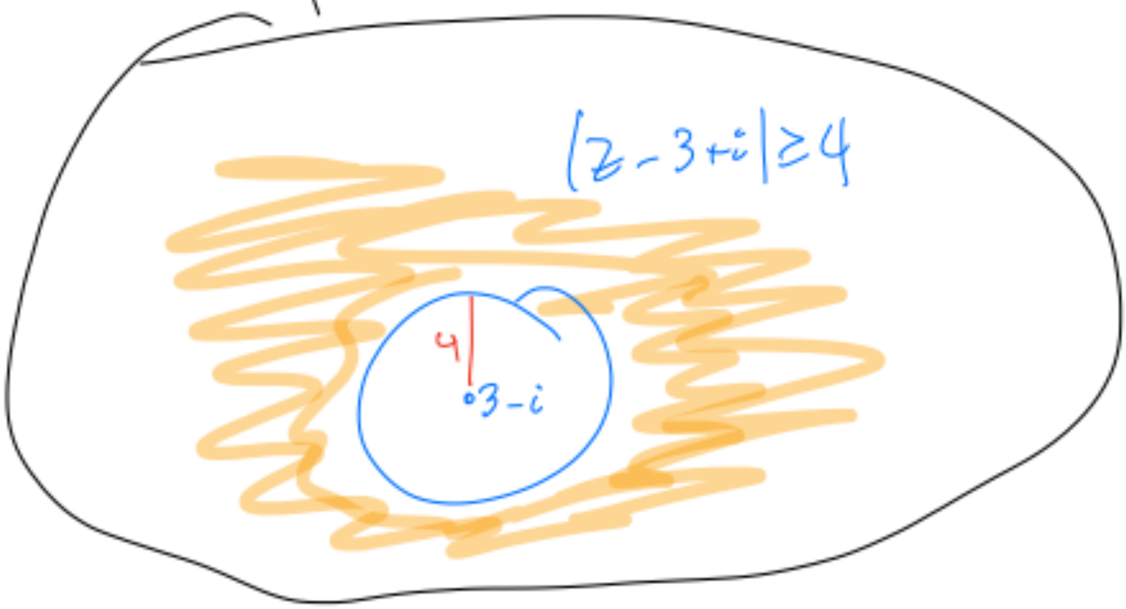
i.e. what is the set $\{z : |z-3+i|=4\}$?

$|z-(3-i)|=4$

$\begin{pmatrix}\text{distance from } z \text{ to} \\ 3-i\end{pmatrix}=4$



circle of radius 4 centered at $3-i$.



Examples of Solving Equations:

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \longleftarrow \text{sometimes these are complex}$$

$$a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=0.$$

Fundamental Theorem of Algebra: every poly. of degree n (with $\mathbb{C}$ coefficients) can be factored completely (with $\mathbb{C}$ roots).

(we will prove this here!)

— Next time we'll see examples.

$$5^{3+2i} = 5^3 \cdot 5^{2i}$$

$$= 125\left(e^{\ln 5}\right)^{2i}$$

$$= 125\left(e^{i(2\ln 5)}\right)$$

$$= 125\left(\cos(2\ln 5) + i\sin(2\ln 5)\right)$$